The Variational Energy Formulation for the Integrated Force Method

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The integrated force method (IFM) is one of the five formulations of mechanics, the others being the flexibility, stiffness, mixed, and total methods. To date, all but the IFM have been associated with variational functionals. A variational functional (VF) has been developed for the IFM. The stationary condition of the VF for the IFM yields the equilibrium and compatibility equations as well as the force and displacement boundary conditions. The stationary condition also yields a new set of boundary equations identified as the boundary compatibility conditions. This paper presents the theory of the variational functional for the IFM. It is illustrated by examples from discrete structures, the plane stress problem of elasticity, and Kirchhoff's plate bending problem. The properties of the VF and its relationship to the potential and complementary energy functionals are shown for discrete analysis.

Nomenclature

A	= variational functional, equilibrium related
a,b	= spans of plate
B	= variational functional, compatibility related
$[B]$	= equilibrium matrix ($m \times n$)
$[B_1]$	= self stress matrix ($n \times r$)
B_x,B_y	= body force components
$[C]$	= compatibility matrix ($r \times n$)
D,E	= plate rigidity and Young's modulus, respectively
DDR	= deformation displacement relationship
dof	= displacement degrees of freedom
E_e	= strain energy
E_c	= complementary strain energy
$\{F\}$	= force vector (n)
fof	= force degrees of freedom
$[G]$	= flexibility matrix ($n \times n$)
h	= plate thickness
I	= moment of inertia
IFM	= integrated force method
$[J]$	= displacement coefficient matrix ($m \times n$)
$[K]$	= stiffness matrix ($m \times m$)
$\mathcal{L}_1,\mathcal{L}_2$	= boundary segments
ℓ	= span of beam
M_x,M_y,M_{xy}	= plate bending moments
M^e	= redundant moment
m	= displacement degrees of freedom
n	= force degrees of freedom
n_x,n_y	= direction cosines
$\{P\}$	= load vector (m)
P_x,P_y,q	= external loads
$\{R\}$	= redundant vector (r)
r	= number of redundants
$[S]$	= IFM matrix ($n \times n$)
SFM	= standard force method
U,V	= potential function for forces
u,v,w	= displacement components
$\bar{u},\bar{v}$	= prescribed displacements
VF	= variational functional
$\{X\}$	= displacement vector (m)
$\{\beta\}$	= deformation vector (n)
$\{\beta_e\}$	= elastic deformation vector (n)
$\{\beta_0\}$	= initial deformation vector (n)
$\{\delta R\}$	= effective initial deformations (r)
δw	= virtual w displacement
$\kappa_x\kappa_y,\kappa_{xy}$	= plate curvatures
ν	= Poisson's ratio
π_c	= potential energy functional
π_p	= complementary energy functional
π_s	= variational functional of IFM
$\sigma_x,\sigma_y,\tau_{xy}$	= plane stress components
σ^e	= redundant stresses
Φ	= Airy's stress function
ψ_1,ψ_2	= moment functions

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Introduction

THE demise of a once popular force method[1-5] in the mid-1960s was not due to generic limitations, but because of the improper formulation of the method. A novel force formulation termed the integrated force method (IFM) has since been developed.[6-9] The IFM bypasses the limitations of the classical force method, henceforth referred to as the standard force method (SFM). The IFM can be identified as one of the five formulations of mechanics, the other four being the SFM, the displacement method (DM), the mixed method (MM),[10] and the total formulation (TF).[11] In the IFM, the equilibrium equations (EE) and the compatibility conditions (CC) are satisfied simultaneously in the force variables. The equations of the IFM can be obtained directly by establishing the EE and CC or generated by following a variational approach. Variational functionals (VF) exist for four of the analysis methods (SFM, DM, MM, and TF). However, until now no VF has been developed for the IFM; this paper presents such a development. The stationary condition of the VF yields the equilibrium equations, the compatibility conditions, and the appropriate boundary conditions of IFM. The VF of the IFM is illustrated by taking examples from discrete structures, plane stress problem of elasticity, and Kirchhoff's plate bending problem.

Basic Theory of the IFM

A structure for the purpose of analysis can be designated as structure (n,m), where n and m are the force and displacement degrees of freedom, respectively. The n component force vector $\{F\}$ must satisfy the m EE along with the $r=(n-m)$ CC. If $n=m$, the structure is determinate and its analysis is trivial. The emphasis here is on the analysis of indeterminate structure for which $n>m$.

The Equilibrium Equations

The EE for discrete analysis can be symbolized as

$$[B]\{F\}=\{P\} \tag{1}$$

Forces are the primal variables (PV) of the equilibrium state. The equilibrium matrix $[B]$ is rectangular and banded in the PV. The potential of internal energy of the structure $E_e=\{X\}^T\{P\}$ can also be written using Eq. (1) as $E_e=\{X\}^T[B]\{F\}$. Thus, for semantics, displacements can be termed as the dual variables of the equilibrium state.

The Compatibility Conditions

The CCs in IFM are generated by eliminating the displacements from the deformation displacement relationships (DDR) of the structure. The DDRs are derived on an energy basis. The internal energy (IE) of discrete structures (n,m) can be expressed in the following forms:

$$\text{IE}=\tfrac{1}{2}\{F\}^T\{\beta\}=\tfrac{1}{2}\{X\}^T\{P\} \tag{2}$$

The IE can be rewritten in terms of internal forces using the EE [Eq. (1)] as

$$\text{IE}=\tfrac{1}{2}\{F\}^T[B]^T\{X\}=\tfrac{1}{2}\{F\}^T\{\beta\} \tag{3a}$$

or

$$\tfrac{1}{2}\{F\}^T([B]^T\{X\}-\{\beta\})=0 \tag{3b}$$

Since the force vector $\{F\}$ is not a null vector,

$$\{\beta\}=[B]^T\{X\} \tag{4}$$

In the DDR given by Eq. (4), the n deformations $\{\beta\}$ are expressed in m displacements $\{X\}$; thus, there are $(n\text{-}m)$ constraints on deformations $\{\beta\}$. The constraints on deformation are the compatibility condition and the CC can be symbolized as

$$[C]\{\beta\}=\{0\} \tag{5}$$

The primal variables of the CC are deformations $\{\beta\}$. In the finite element analysis, elemental deformations have to be compatible to the neighboring elements only; thus, the CC are banded. The maximum bandwidth (MBW) of the CC of an element depends on the force degrees of freedom of its neighboring elements.

For illustration, we take the examples of a plate and a truss shown in Fig. 1. The force degrees of freedom (fof) for the plate element is fof = 9 and the truss element is fof = 1. The MBW of compatibility matrix $[C]$ are

Plate: interior element $(\text{MBW})_{\text{Zone }1}=81$

boundary element $(\text{MBW})_{\text{Zone }2}=54$

corner element $(\text{MBW})_{\text{Zone }3}=36$

Truss: interior element $(\text{MBW})_{\text{Zone }1}=16$

boundary element $(\text{MBW})_{\text{Zone }2}=6$

The $(r\times n)$ compatibility matrix $[C]$ of discrete analysis is independent of material properties, design parameters, and loads; also, it is rectangular and banded. The compatibility matrix is generated by eliminating the displacements from the DDR utilizing the MBW information as reported in Ref. 8.

The generation of CC essentially consists of the following two steps:

1) Pick any deformation, e.g., β_1. Establish the bandwidth for β_1 and let these deformations constitute the set S_1. Eliminate displacements from DDR of set S_1 to generate the first CC. Reduce the number of DDR by one, from n to $(n-1)$ by dropping one DDR participating in the CC generated.

2) For the $(n-1)$ DDR, follow step 1 to generate the second CC. Repeat the step until all of the CC are generated.

The compatibility matrix $[C]$ satisfies the equilibrium state[9]

$$[B][C]^T=[0] \tag{6}$$

The condition given by Eq. (6) can be shown by eliminating deformation $\{\beta\}$ between Eqs. (5) and (4) as

$$[C][B]^T\{X\}=\{X\}^T([B][C]^T)=\{0\} \tag{7}$$

Since the displacement $\{X\}$ is not a null vector, $[B][C]^T=[0]$.

In summary, it can be said that both the EE matrix $[B]$ and CC matrix $[C]$ in their respective primal variables are rectangular, banded, nonsymmetric and independent of the material properties, design parameters, and external loads. The self-stress matrix $[B_1]$ of the SFM can be related to the compatibility matrix $[C]$ as $[B_1]=[C]^T$, since both the matrices satisfy the equilibrium state given by Eq. (6).

The total deformation can be due to mechanical loads $\{\beta_e\}$, thermal expansion, initial misfit of elements $\{\beta_0\}$, etc.,

$$\{\beta\}=\{\beta_e\}+\{\beta_0\} \tag{8}$$

The CC can be written in elastic deformation $\{\beta_e\}$ as

$$[C]\{\beta\}=[C]\{\beta_e\}+[C]\{\beta_0\}=\{0\}$$

$$[C]\{\beta_e\}=-[C]\{\beta_0\}=\{\delta R\} \tag{9}$$

Even though any element can contribute to the n component initial deformation vector $\{\beta_0\}$, only the $r=(n\text{-}m)$th-order effective initial deformation (EID) vector $\{\delta R\}$ induces stresses in the structure. The potential of the internal forces due to initial deformations can be written in terms of EID and the redundant forces $\{R\}$ of the standard force method as

$$E_c=\{R\}^T\{\delta R\}=-\{R\}^T[C]\{\beta_0\} \tag{10}$$

From the energy scalar [Eq. (10)], the dual variables of the CC are the redundant forces. The EE and CC in their dual variables give rise to symmetric matrices $[K]=[B]^T[G]^{-1}[B]$ and $[Q]=[C][G][C]^T$, respectively.

Analysis of Indeterminate Structure by IFM

In the IFM, we write the compatibility conditions in the force variables using the deformation force relation as

$$\{\beta_e\}=[G]\{F\} \tag{11}$$

or

$$[C]\{\beta_e\}=[C][G]\{F\}=\{\delta R\} \tag{12}$$

Coupling of the CC [Eq. (12)] to the EE [Eq. (1)] yields the governing equations of the integrated force method as

$$\begin{bmatrix} [B] \\ \hline [C]\,[G] \end{bmatrix}\{F\}=\left\{\begin{array}{c} P \\ \hline \delta R \end{array}\right\} \quad \text{or} \quad [S]\{F\}=\{P\}^* \tag{13}$$

The matrix $[S]$ of the IFM is assembled without recourse to the redundants or the basis determinate structure of the SFM. Once the forces are known from Eq. (1), the

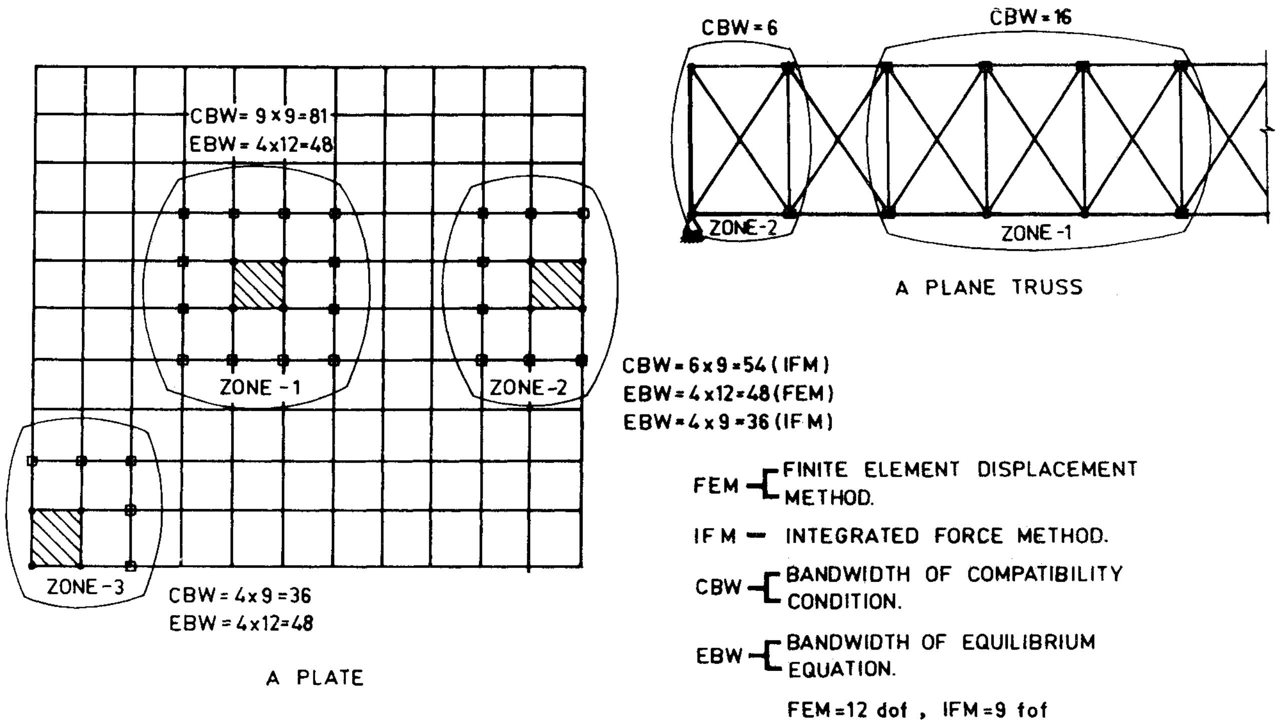

Fig. 1 Bandwidth of compatibility condition.

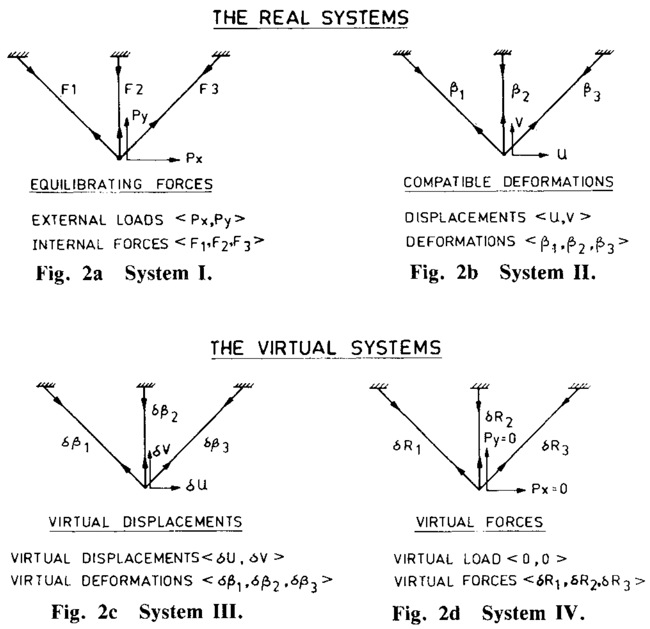

Fig. 2a System I. **Fig. 2b System II.** **Fig. 2c System III.** **Fig. 2d System IV.**

displacements can be calculated using the following formula given in Refs. 7 and 8:

$$\{X\} = [J]\,[G]\,\{F\} \tag{14}$$

The Variational Energy Functional

The force method of analysis (irrespective of IFM or SFM) requires both the EE and CC together. There exists no variational functional from which both the EE and CC are obtained simultaneously. The proposed variational formulation yields both EE and CC together. For simplicity, the variational functional is derived from the principle of virtual work, taking the example of a three-bar truss, and then generalized. The three-bar truss is shown in Fig. 2. The element forces (F_1, F_2, and F_3) are in equilibrium with the external loads P_x and P_y (Fig 2a, system I). The element deformations (β_1, β_2, and β_3) are compatible with the nodal displacements u, v (Fig. 2b, system II). Figure 2c (system III) shows the virtual displacements δu and δv and corresponding compatible virtual deformations $\delta\beta_1$, $\delta\beta_2$, and $\delta\beta_3$. Figure 2d (system IV) depicts the self-equilibrating virtual forces δR_1, δR_2, and δR_3.

According to the principle of virtual work, the virtual work done by real forces (system I) undergoing virtual deformations (system III) is zero,

$$F_1\delta\beta_1 + F_2\delta\beta_2 + F_3\delta\beta_3 - (P_x\delta u + P_y\delta v) = 0 \tag{15}$$

From the principle of complementary virtual work, the virtual work done by real deformations (system II) and virtual forces (system IV) is zero,

$$\beta_1\delta R_1 + \beta_2\delta R_2 + \beta_3\delta R_3 = 0 \tag{16}$$

Equations (15) and (16) can be added to obtain the variational function of the IFM as

$$\delta\pi_s = (F_1\delta\beta_1 + F_2\delta\beta_2 + F_3\delta\beta_3) + (\beta_1\delta R_1 + \beta_2\delta R_2 + \beta_3\delta R_3)$$
$$- (P_x\delta u + P_y\delta v) = 0$$
$$= \delta A + \delta B - \delta W = 0 \tag{17}$$

or

$$\pi_s = A + B - W \tag{18}$$

The functional A and W are associated with the equilibrium of the structure. The variables of the functionals A and W are three deformations (β_1,β_2,β_3) and two displacements (u,v). The deformations in functional $A(F,\beta)$ are changed to displacements using the displacement deformation relationship. The functional $A(F,u)$ has the following explicit form:

$$A(F,u) = \frac{F_1(v-u)}{\sqrt{2}} + F_2 V + \frac{F_3(u+v)}{\sqrt{2}} \tag{19a}$$

For two-dimensional elasticity problem, the functional $A(\sigma,u)$ has the following explicit form:

$$A(\sigma,u)=h\int_S\left[\sigma_x\frac{\partial u}{\partial x}+\sigma_y\frac{\partial v}{\partial y}+\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\mathrm{d}x\mathrm{d}y \tag{19b}$$

The functional $A(\sigma,u)$ represents the strain energy in which the stresses and displacements are treated independently.

The functional $B(\beta,R)$ assures the compatibility of the structural deformations. For two-dimensional elasticity, the functional $B(\epsilon,\sigma^e)$ can be written as

$$B(\epsilon,\sigma^e)=h\int_S(\epsilon_x\sigma_x^e+\epsilon_y\sigma_y^e+\gamma_{xy}\tau_{xy}^e)\mathrm{d}x\mathrm{d}y \tag{20}$$

The functional $B(\epsilon,\sigma^e)$ is the complementary strain energy functional in which the strains ϵ and redundant stresses σ^e are treated independently. For the IFM, the strains are to be replaced by stresses using stress-strain law. The functional $B(F,R)$ for the three-bar truss has the following explicit form:

$$B(F,R)=\left(\frac{\ell\sqrt{2}}{E}\right)\left(\frac{F_1}{A_1}-\frac{F_2}{A_2}+\frac{F_3}{A_3}\right)R \tag{21}$$

The functional $B(\sigma,\sigma^e)$ for two-dimensional elasticity problem can be written as

$$B(\sigma,\sigma^e)=h\int_S\left[\frac{(\sigma_x-\nu\sigma_y)}{E}\sigma_x^e+\frac{(\sigma_y-\nu\sigma_x)}{E}\sigma_y^e+\frac{1+\nu}{E}\tau_{xy}\tau_{xy}^e\right]\mathrm{d}x\mathrm{d}y \tag{22}$$

The potential of the external forces $W(P,u)$ has three components,

$$W(P,u)=h\int_S(B_xu+B_yv)\mathrm{d}x\mathrm{d}y+\int_{\mathcal{L}_1}(\bar{P}_xu+\bar{P}_yv)\mathrm{d}\ell_1+\int_{\mathcal{L}_2}(P_x\bar{u}+P_y\bar{v})\mathrm{d}\ell_2 \tag{23}$$

The first integral in Eq. (23) is due to body forces B_x and B_y, the second is for the portion of the boundary $\mathcal{L}_1$ on which the external loads $\bar{P}_x$ and $\bar{P}_y$ are specified, and the remaining term is for the boundary $\mathcal{L}_2$ on which displacements $\bar{u}$ and $\bar{v}$ are prescribed.

The variational functional $\pi_s(\sigma,u,\sigma^e)$ of the IFM is obtained by adding the three expressions: $\pi_s=A+B-W$. The variables of the functional π_s for variational purpose are the displacements u and redundant stresses σ^e.

Three illustrative examples show the derivation of the IFM formulations from the stationary condition of the functional π_s.

Example 1: Beltrami-Michell Formulation

The IFM of plane elasticity problem is the Beltrami-Michell formulation (BMF).[12] The BMF has two EEs and one CC in the field expressed in stresses. The variational variables of the functional π_s^{BMF} are two displacements u and v and one redundant stress function. The latter is taken as the Airy's stress function Φ as

$$\sigma_x^e=\frac{\partial^2\phi}{\partial y^2}-V;\quad\sigma_y^e=\frac{\partial^2\phi}{\partial x^2}-V;\quad\tau_{xy}^e=-\frac{\partial^2\phi}{\partial x\partial y} \tag{24}$$

The VF of the BMF has the following form:

$$\begin{aligned}\pi_s^{\mathrm{BMF}}&=h\int_S\left[\sigma_x\frac{\partial u}{\partial x}+\sigma_y\frac{\partial v}{\partial y}+\tau_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\mathrm{d}x\mathrm{d}y\\&+h\int_S\left[\frac{(\sigma_x-\nu\sigma_y)}{E}\left(\frac{\partial^2\phi}{\partial y^2}-V\right)+\frac{(\sigma_y-\nu\sigma_x)}{E}\left(\frac{\partial^2\phi}{\partial x^2}-V\right)\right.\\&\left.-2\frac{(1+\nu)}{E}\tau_{xy}\frac{\partial^2\phi}{\partial x\partial y}\right]\mathrm{d}x\mathrm{d}y-h\int_S(B_xu+B_yv)\mathrm{d}x\mathrm{d}y\\&-\int_{\mathcal{L}_1}(\bar{P}_xu+\bar{P}_yv)\mathrm{d}\ell_1-\int_{\mathcal{L}_2}(P_x\bar{u}+P_y\bar{v})\mathrm{d}\ell_2\end{aligned} \tag{25}$$

The stationary condition of the VF after algebraic simplification can be written as

$$\begin{aligned}\delta\pi_s^{\mathrm{BMF}}&=h\left\{\int_S\left[\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+B_x\right)\delta u\right.\right.\\&\left.+\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+B_y\right)\delta v\right]\mathrm{d}s+\frac{1}{E}\int_S\left[\frac{\partial^2}{\partial x^2}(\sigma_y-\nu\sigma_x)\right.\\&\left.\left.+\frac{\partial^2}{\partial y^2}(\sigma_x-\nu\sigma_y)-2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\partial y}\right]\delta\phi\mathrm{d}s\right\}\\&-h\int_\ell\left\{\left[(\sigma_xn_x+\tau_{xy}n_y-\bar{P}_x)\delta u+(\tau_{xy}n_x+\sigma_yn_y-\bar{P}_y)\delta v\right]\right.\\&+[(u-\bar{u})\delta(\sigma_xn_x+\tau_{xy}n_y)+(v-\bar{v})\delta(\tau_{xy}n_x+\sigma_yn_y)]\\&+\frac{1}{E}\left[\frac{\partial}{\partial x}(\sigma_y-\nu\sigma_x)n_x+\frac{\partial}{\partial y}(\sigma_x-\nu\sigma_y)n_y\right.\\&\left.\left.-(1+\nu)\left(\frac{\partial\tau_{xy}}{\partial x}n_y+\frac{\partial\tau_{xy}}{\partial y}n_x\right)\right]\delta\phi\right\}\mathrm{d}\ell\end{aligned} \tag{26}$$

Since u, v, and Φ are nonzero quantities, their coefficients in Eq. (26) have to be zero, which leads to the following field equations and boundary conditions:

1) The field equilibrium equations are

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+B_x=0\quad\text{and}\quad\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+B_y=0 \tag{27}$$

2) The field compatibility condition has the form

$$\frac{\partial^2}{\partial x^2}(\sigma_y-\nu\sigma_x)+\frac{\partial^2}{\partial y^2}(\sigma_x-\nu\sigma_y)-2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\partial y}=0 \tag{28}$$

The CC given by Eq. (28) can be simplified using the EE [Eq. (27)] and relationship between body forces and its potential $B_x=\partial V/\partial x$ and $B_y=\partial V/\partial y$ as

$$\nabla^2(\sigma_x+\sigma_y)+(1+\nu)\nabla^2V=0 \tag{29}$$

The line integrals of Eq. (26) provide the following boundary conditions:

3) Along the boundary where the forces are prescribed as shown in Fig. 3, we have the following stress boundary conditions as the coefficients of δu, δv, and $\delta\phi$ in the line integrals:

$$\sigma_xn_x+\tau_{xy}n_y=\bar{P}_x\quad\text{and}\quad\tau_{xy}n_x+\sigma_yn_y=\bar{P}_y \tag{30}$$

$$\frac{\partial}{\partial x}(\sigma_y-\nu\sigma_x)n_x+\frac{\partial}{\partial y}(\sigma_x-\nu\sigma_y)n_y-(1+\nu)\left(\frac{\partial\tau_{xy}}{\partial x}n_y+\frac{\partial\tau_{xy}}{\partial y}n_x\right)=0 \tag{31}$$

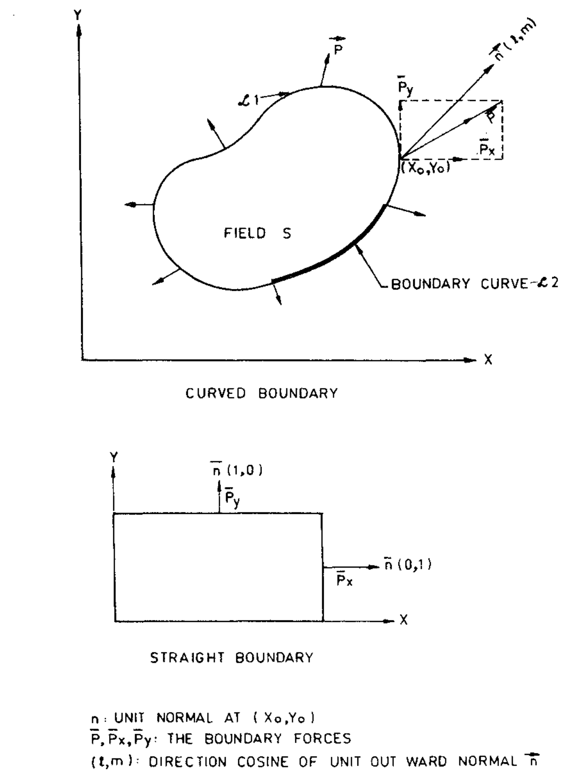

Fig. 3 First boundary value problem of elasticity.

The boundary conditions [Eq. (30)] are those of the classical stress in the plane elasticity problem. The boundary condition (BC) given by Eq. (31) is identified as to be the novel boundary compatibility condition.

4) Along the boundary where displacements are prescribed, we have the displacement boundary condition as the coefficients of $\delta\bar{P}_x$ and $\delta\bar{P}_y$ in the line integral,

$$u=\bar{u} \text{ and } v=\bar{v} \tag{32}$$

The expressions given by Eqs. (27-32) derived from the VF constitute the IFM for the plane stress problem.

Example 2: Plate Bending Problem

As an illustration from continuum mechanics, the Kirchhoff's plate formulation[13] is derived from the stationary condition of the VF for the IFM. The variational variables of the problem are the transverse displacement w and the two moment functions ψ_1 and ψ_2 defined as

$$M_x^e=\frac{\partial\psi_1}{\partial y}+U \tag{33a}$$

$$M_y^e=\frac{\partial\psi_2}{\partial x}+U \tag{33b}$$

$$M_{xy}^e=-\frac{1}{2}\left(\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}\right) \tag{33c}$$

The moment functions ψ_1 and ψ_2 satisfy the plate equilibrium equations, provided the distributed transverse load q is linked to the potential function U as

$$q=\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=\nabla^2 U \tag{34}$$

The plate curvature moment relationships are as follows:

$$\kappa_x=\frac{\partial^2 w}{\partial x^2}=\frac{(M_x-\nu M_y)}{D'} \tag{35a}$$

$$\kappa_y=\frac{\partial^2 w}{\partial y^2}=\frac{(M_y-\nu M_x)}{D'} \tag{35b}$$

$$\kappa_{xy}=\frac{\partial^2 w}{\partial x\partial y}=\frac{(1+\nu)M_{xy}}{D'} \tag{35c}$$

where $D'=Eh^3/12$.

The variational functional of the problem has the following explicit form:

$$\begin{aligned}\pi_s^P=&\int_S\left(M_x\frac{\partial^2 w}{\partial x^2}+M_y\frac{\partial^2 w}{\partial y^2}+2M_{xy}\frac{\partial^2 w}{\partial x\partial y}\right)\mathrm{d}x\mathrm{d}y\\&+\int_S\left[\left(\frac{\partial\psi_1}{\partial y}+U\right)\left(\frac{M_x-\nu M_y}{D'}\right)+\left(\frac{\partial\psi_2}{\partial x}+U\right)\left(\frac{M_y-\nu M_x}{D'}\right)\right.\\&\left.-\left(\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_2}{\partial y}\right)\frac{(1+\nu)}{D'}M_{xy}\right]\mathrm{d}x\mathrm{d}y-\int_S qw\mathrm{d}x\mathrm{d}y\end{aligned} \tag{36}$$

The stationary condition of π_s^P with respect to the independent variables (w, ψ_1, and ψ_2) can be written as

$$\begin{aligned}\delta\pi_s^P=&\int_S\left(\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}-q\right)\delta w\mathrm{d}x\mathrm{d}y\\&+\int_S\frac{1}{D'}\left\{\left[(1+\nu)\frac{\partial M_{xy}}{\partial x}-\frac{\partial}{\partial y}(M_x-\nu M_y)\right]\delta\psi_1\right.\\&\left.+\left[(1+\nu)\frac{\partial M_{xy}}{\partial y}-\frac{\partial}{\partial x}(M_y-\nu M_x)\right]\delta\psi_2\right\}\mathrm{d}x\mathrm{d}y\\&-\int_{\mathcal{L}}\left(\frac{\partial M_x}{\partial x}n_x+\frac{\partial M_y}{\partial y}n_y+2\frac{\partial M_{xy}}{\partial x}n_y+2\frac{\partial M_{xy}}{\partial y}n_x\right)\delta w\mathrm{d}\ell\\&+\frac{1}{D'}\int_{\mathcal{L}}\left\{\left[(M_x-\nu M_y)n_y-(1+\nu)M_{xy}n_x\right]\delta\psi_1\right.\\&\left.+[(M_y-\nu M_x)n_x-(1+\nu)M_{xy}n_y]\delta\psi_2\right\}\mathrm{d}\ell=0\end{aligned} \tag{37}$$

1) The field equilibrium equation is associated with the variational parameter δw in the surface integral

$$\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_y}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}=q \tag{38}$$

2) The field compatibility conditions are associated with the variational parameters $\delta\psi_1$ and $\delta\psi_2$ in the surface integrals as

$$\frac{\partial}{\partial x}(M_y-\nu M_x)-(1+\nu)\frac{\partial M_{xy}}{\partial y}=0 \tag{39}$$

$$\frac{\partial}{\partial y}(M_x-\nu M_y)-(1+\nu)\frac{\partial M_{xy}}{\partial x}=0 \tag{40}$$

The line integrals in Eq. (37) yield the following boundary conditions for the rectangular plate shown in Fig. 4:

3) The coefficient of δw in the line integral is as

$$\begin{aligned}&\left\{\left[\left(\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}\right)+\frac{\partial M_{xy}}{\partial y}\right]n_x\right.\\&\left.+\left[\left(\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}\right)+\frac{\partial M_{xy}}{\partial x}\right]n_y\right\}=0\end{aligned} \tag{41}$$

The BC given by Eq. (41) for the rectangular plate can be written as

$$\text{Along } x=0 \text{ and } x=a, \ w=\text{prescribed or } Q_x+\frac{\partial M_{xy}}{\partial y}=0$$

$$\text{Along } y=0 \text{ and } y=b, \ w=\text{prescribed or } Q_y+\frac{\partial M_{xy}}{\partial x}=0 \tag{42}$$

$$Q_x=\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y} \text{ and } Q_y=\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x} \tag{43}$$

4) The coefficients of rotations $\delta(\partial w/\partial x)$ and $\delta(\partial w/\partial y)$ in the line integral yield the following BC:

$$\text{Along } x=0 \text{ and } x=a, \ \frac{\partial w}{\partial x} \text{ is prescribed or } M_x=0 \tag{44}$$

$$\text{Along } y=0 \text{ and } y=b, \ \frac{\partial w}{\partial y} \text{ is prescribed or } M_y=0 \tag{45}$$

5) The coefficients of the stress functions $\delta\psi_1$ and $\delta\psi_2$ yield the following two boundary conditions:

$$(M_x-\nu M_y)n_y-(1+\nu)M_{xy}n_x=0 \tag{46}$$

$$(M_y-\nu M_x)n_x-(1+\nu)M_{xy}n_y=0 \tag{47}$$

The BC given by Eqs. (46) and (47) are identified to be the novel boundary compatibility conditions that must be satisfied along the portion of the boundary where the moments are prescribed. The expressions given by Eqs. (38-47) constitute the IFM for the plate bending problem.

Example 3: A Propped Beam

The VF of discrete structures consisting of beam elements can be obtained from Eq. (36) of plate with the following simplifications:

$$M_x=M, \quad M_y=M_{xy}=0, \quad \psi_1=M^e, \quad \psi_2=0,$$

$$D_1=\mathrm{EI}, \quad dy=1, \quad ds=d\ell$$

$$\pi_s^b=\int_{\mathcal{L}}\left(M\frac{\partial^2 w}{\partial x^2}\right)d\ell+\int_{\mathcal{L}}\frac{MM^e}{\mathrm{EI}}d\ell-\int_{\mathcal{L}}qw\,d\ell \tag{48}$$

The VF of the IFM for discrete structures is illustrated by examples of the propped beam shown in Fig. 5. The structure is idealized by two beam elements. Each element has two force (M_1,M_2) and four displacement $(w_1,w_2,\theta_1,\theta_2)$ degrees of freedom. The master fof is (M_1,M_2,M_3) and the dof (w,θ,β) is as shown in Fig. 5. The variables of the final equations of the structure are moments. For each element, the moments are assumed to be linear in the span, using the linear Hermite interpolation polynomials as

$$M=H_{01}^{\circ}(x)M_1+H_{02}^{\circ}(x)M_2 \tag{49}$$

where

$$H_{01}^{\circ}(x)=-(x-\ell)/\ell \text{ and } H_{02}^{\circ}(x)=x/\ell$$

For a beam, the cubic displacement fields are assumed in Hermite polynomials as

$$w(x)=H_{01}'(x)w_1+H_{02}'(x)w_2+H_{11}'(x)\theta_1+H_{12}'(x)\theta_2 \tag{50}$$

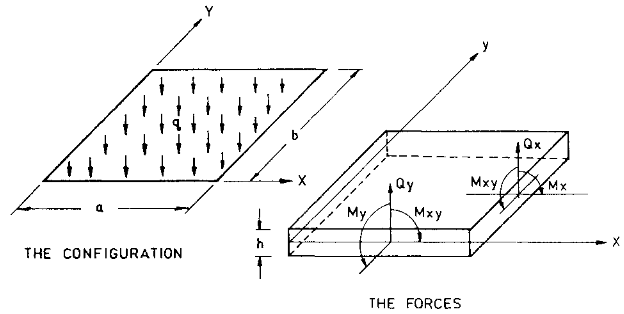

Fig. 4 Kirchhoff's plate bending problem.

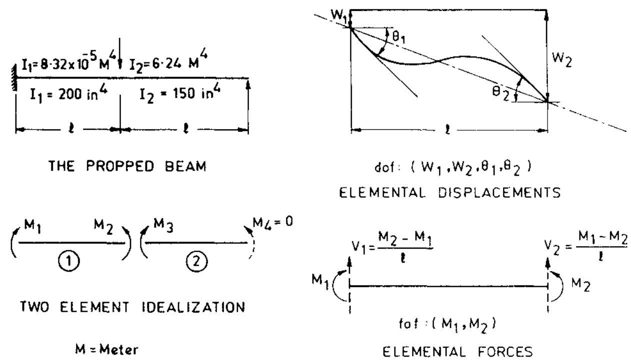

Fig. 5 Analysis of propped beam.

where:

$$H_{01}'(x)=\frac{1}{\ell^3}(3x^3-2\ell x^2+\ell^3)$$

$$H_{02}'(x)=\frac{-1}{\ell^3}(2x^3-3\ell x^2)$$

$$H_{11}'(x)=\frac{1}{\ell^2}(x^3-2\ell x^2+\ell^2 x)$$

$$H_{12}'(x)=\frac{1}{\ell^2}(x^3-2\ell x^2)$$

The first term in the VF can be discretized using the moment and displacement functions as

$$\text{Term } 1=A(X,F)=\{X\}^T[B]\{F\}$$

$$=\langle w_1,w_2,\theta_1,\theta_2\rangle\begin{bmatrix}-1/\ell & 1/\ell\\ 1/\ell & -1/\ell\\ -1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}M_1\\ M_2\end{Bmatrix} \tag{51}$$

To discretize the second term, the variation of redundant moment is assumed to be identical to the variation of primary moment,

$$M^e=R(H_{01}^{\circ}M_1^e+H_{02}^{\circ}M_2^e) \tag{52}$$

Substitution of M and M^e in the second term of the VF and upon integration, we have the following discretized energy term:

$$\text{Term } 2=B(M^e,M)=R\langle M_1^e M_2^e\rangle\left(\frac{\ell}{6\mathrm{EI}}\right)\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}M_1\\ M_2\end{Bmatrix} \tag{53}$$

The coefficient matrix in Eq. (53) can be identified as the flexibility matrix $[G]$ of the beam element. The equation can then be rewritten as

$$B(M^e,M)=R\langle M^e\rangle[G]\{M\} \tag{54}$$

From the discretized functionals A and B, the VF of the structure π_s^b can be constructed by correlating the elemental degrees of freedom to the master degrees of freedom as

$$\pi_s^b=\left\{\frac{(M_1-M_2)}{\ell}w+(M_2-M_3)\theta-\frac{M_3}{\ell}w\right.$$
$$\left.+R\left\{M_1^e\left(\frac{2M_1+M_2}{6\text{EI}}\right)+M_2^e\left(\frac{2M_2+M_1}{6\text{EI}}\right)+\frac{2M_3^eM_3}{6\text{EI}_2}-Pw\right.\right. \tag{55}$$

The stationary condition of the VF with respect to (w,θ,R) can be written as

$$\delta\pi_s^b=\frac{\partial\pi_s^b}{\partial w}\delta w+\frac{\partial\pi_s^b}{\partial\theta}\delta\theta+\frac{\partial\pi_s^b}{\partial R}\delta R=0 \tag{56}$$

Since δw, $\delta\theta$, and δR are nonzero quantities, $\delta\pi_s^b=0$ yields

$$\frac{\partial\pi_s^b}{\partial w}=\frac{\partial\pi_s^b}{\partial\theta}=\frac{\partial\pi_s^b}{\partial R}=0 \tag{57}$$

Equation (57) has the following explicit form:

$$\begin{bmatrix}\frac{1}{\ell} & -\frac{1}{\ell} & -\frac{1}{\ell}\\ 0 & 1 & -1\\ \frac{\ell(2M_1^e+M_2^e)}{6\text{EI}_1} & \frac{\ell(M_1^e+2M_2^e)}{6\text{EI}_1} & \frac{\ell M_3^e}{3\text{EI}_2}\end{bmatrix}\begin{Bmatrix}M_1\\ M_2\\ M_3\end{Bmatrix}=\begin{Bmatrix}P\\ 0\\ 0\end{Bmatrix} \tag{58}$$

The values for M_1^e,M_2^e,M_3^e can be obtained by eliminating the displacements w and θ from the deformation displacement relations of the structure given by

$$\{\beta\}=[B]^T\{X\}$$

$$\begin{Bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{Bmatrix}=\begin{bmatrix}1/\ell & 0\\ -1/\ell & 1\\ -1/\ell & -1\end{bmatrix}\begin{Bmatrix}w\\ \theta\end{Bmatrix} \tag{59}$$

The compatibility condition for the propped beam from Eq. (59) can be obtained by eliminating the displacements w and θ as

$$(2\beta_1+\beta_2+\beta_3)=0 \tag{60}$$

or

$$\langle M_1^e,M_2^e,M_3^e\rangle=\langle 2,1,1\rangle \tag{61}$$

The explicit form of the equations of the IFM can be written as

$$\begin{bmatrix}1.0 & -1.0 & -1.0\\ 0.0 & 1.0 & -1.0\\ 1.25 & 1.0 & 0.67\end{bmatrix}\begin{Bmatrix}M_1\\ M_2\\ M_3\end{Bmatrix}=\begin{Bmatrix}P\ell\\ 0\\ 0\end{Bmatrix} \tag{62}$$

Equation (62) represents the formulation by the IFM for the propped beam.

Other Variational Formulations

Structural analysis methods can be divided into five broad categories, depending on the choice of primary unknowns. The formulations and their VF are: 1) the total formulation (Washizu variational functional), 2) the displacement method (potential energy functional), 3) the integrated force method (the VF for the IFM), 4) the standard force method (the complementary energy functional), and 5) the mixed method (Reissner's functional). The formulations along with their VF are depicted in Table 1.

The five variational functions for discrete systems in matrix notation are presented in Table 2. The stationary condition of the functionals π_p, π_c, π_s, π_R, and π_w yield the EE in displacements, CC in redundants, EE and CC in forces, EE in forces with the deformation force relationship, and EE in forces and the deformation force relationship along with DDR, respectively. The VF for the IFM for discrete system can be constructed as indicated for the beam in example 3.

Properties of VF for IFM

The properties of the VF for IFM are shown for discrete analysis. The following expressions are required to examine the properties of the VF of IFM.

1) The equilibrium state of the structure can be written in forces or displacements as

$$[B]\{F\}=\{P\}\quad\text{or}\quad[K]\{X\}=\{P\} \tag{63}$$

2) The compatibility condition of the structure can be written in deformations or forces as

$$[C]\{\beta\}=\{0\}\quad\text{or}\quad[C][G]\{F\}=\{0\} \tag{64}$$

The compatibility matrix satisfies the equilibrium state $[B][C]^T=[0]$.

3) The internal energy of the structure can be expressed as

$$\text{IE}=\tfrac{1}{2}\{F\}^T[G]\{F\}=\tfrac{1}{2}\{X\}^T[K]\{X\}=\tfrac{1}{2}\{X\}^T\{P\} \tag{65}$$

From Eqs. (1), (4), (11), and (63-65) and the basic relationship of the displacement method, the following identities can be shown:

$$\text{IE}=\tfrac{1}{2}\{X\}^T[B]\{F\} \tag{66}$$

$$\{\beta\}=[G]\{F\}=[B]^T\{X\} \tag{67}$$

$$[K]=[B][G]^{-1}[B]^T;\quad[G]=[B]^T[K]^{-1}[B] \tag{68}$$

Property 1

From Eqs. (63-68), it can be shown that the value of the VF for the IFM at the stationary point (SP) is zero,

$$\begin{aligned}\pi_s|_{\text{SP}}&=\{X\}^T[B]\{F\}+\{R\}^T[C][G]\{F\}-\{X\}^T\{P\}\\ &=\{X\}^T\{P\}+\{R\}^T[C][B]^T\{X\}-\{X\}^T\{P\}\\ &=\{X\}^T[B][C]^T\{R\}=0\end{aligned} \tag{69}$$

Property 2

The potential energy functional π_p can be generated from the VF for the IFM if forces in π_s are eliminated in favor of

Table 1 Methods of structural analysis and associated energy formulation

No.	Name of method: Elasticity	Name of method: Structures	Variables of the methods: Elasticity	Variables of the methods: Structures	Associated energy formulation
1	Navier method	Displacement method	Displacement functions U_i	Deflection vector $\{U\}$	Potential energy (min condition)
2	Airy formulation	Force method	Airy's stress function	Redundant forces $\{R\}$	Complementary energy (max condition)
3	Beltrami-Michell formulation	Integrated force method	Stress σ_{ij}	Forces $\{F\}$	VF of IFM (stationary condition)
4	Mixed method	Reissner method	σ_{ij}, U_i	$\{F\}$ and $\{U\}$	Reissner energy (stationary condition)
5	Total formulation	Washizu method	σ_{ij}, U_i, and strain ϵ_{ij}	$\{F\}$, $\{U\}$, and deformation vector $\{\beta\}$	Washizu energy (min condition)

Table 2 Variational functional for discrete systems

No.	Method	Variational functional	Remarks
1	Displacement	$\pi_p = \frac{1}{2}\{X\}^T[K]\{X\} - \{X\}^T\{P\}$	
2	Standard force	$\pi_c = -\frac{1}{2}\begin{Bmatrix} F^\circ \\ R \end{Bmatrix}^T [G] \begin{Bmatrix} F^\circ \\ R \end{Bmatrix} + \{X\}^T\{P\}$	$\{F^\circ\} = [B_0]^{-1}\{P\}$
3	Integrated force	$\pi_s = \{X\}^T[B]\{F\} + \{R\}^T[C][G]\{F\} - \{X\}^T\{P\}$	
4	Reissner	$\pi_R = \{X\}^T[B]\{F\} - \frac{1}{2}\{F\}^T[G]\{F\} - \{X\}^T\{P\}$	$[B]^T\{X\} = \{\beta\}$
5	Washizu	$\pi_w = \frac{1}{2}\{\beta\}^T[G]^{-1}\{\beta\} + \{X\}^T[B]\{F\} - \{\beta\}^T\{F\} - \{X\}^T\{P\}$	

displacements as

$$\begin{aligned}\pi_s &= \int\delta\pi_s|_{F\to X} = \int\delta(\{X\}^T[B]\{F\}) + \int\delta(\{R\}^T[C][G]\{F\}) \\ &\quad - \int\delta(\{X\}^T\{P\}) \\ &= \int\delta(\{X\}^T[K]\{X\}) + \int\delta(\{R\}^T[C][B]^T\{X\}) \\ &\quad - \int\delta(\{X\}^T\{P\}) \\ &= \tfrac{1}{2}\{X\}^T[K]\{X\} - \{X\}^T\{P\} = \pi_p \end{aligned} \tag{70}$$

or

$$\pi_s|_{F-X} = \pi_p$$

Property 3

The complementary energy functional π_c of the SFM can be generated as a special case of π_s by introducing the concept of redundant forces explicitly as

$$\{F\} = \{F^\circ\} + [C]^T\{R\}; \ \{F^\circ\} = [B_0]^{-1}\{P\}; \ [B] = [B_0 \vdots B_c] \tag{71}$$

$$\begin{aligned}\pi_s|_{F\to R\to X} &= \int\delta\{X\}^T[B]\{F\} + \int\delta\{R\}^T[C][G]\{F\} \\ &\quad - \int\delta\{X\}^T\{P\} \\ &= \int\delta\{X\}^T[[B]\{F\} - \{P\}] \\ &\quad + \int\delta\{R\}^T[C][G](\{F^\circ\} + [C]^T\{R\}) \\ &= \tfrac{1}{2}(\{R\}^T[C][G][C]^T\{R\} \\ &\quad + 2\{R\}^T[C][G]\{F^\circ\} + C_0 \end{aligned} \tag{72}$$

The constant C_0 is independent of redundants $\{R\}$. If the constant C_0 is defined as

$$C_0 = \{F^\circ\}^T[G]\{F^\circ\} - w \tag{73}$$

Then Eq. (72) can be written as:

$$\pi_s|_{F-R-X} = \tfrac{1}{2}\{F\}^T[G]\{F\} - w = \pi_c \tag{74}$$

The expression given by Eq. (74) is the complementary energy functional of the SFM.

Conclusions

The conclusions of this study are as follows:

1) The variational functional of the integrated force method has been developed.

2) The variational functional of IFM is the only functional whose stationary condition simultaneously yields both the equilibrium and compatibility conditions of the structure.

3) The VF of the IFM yields the boundary compatibility conditions.

4) For discrete analysis, the VF of the IFM allows analysts the freedom to choose both force and displacement interpolation functions.

5) From the VF for the IFM the two key functionals (namely, the potential energy functional π_p and the complementary energy functional π_c) can be derived.

6) The value of the VF of the IFM is zero at the stationary point.

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